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PROGRESS REPORT.

The following gentlemen have been elected members of our Society.—

(1) Mr. T. V. Subba Rao, B.A., L.T., Mathematics Assistant, Madras Christian College, 295, Thumbu Chetty Street, Madras (at concessional rate).

(2) Mr. K. B. Wagle, M.A., Accountant General, Allahabad.

(3) Mr. N. B. Macmillan, B. Sc., Professor of Mechanical Engineering, College of Engineering, Poona.

(4) Mr. A. N. Raghavachar, B.A , Student, 909, 4th Main Road, Chamarajapet, Bangalore City (at concessional rate).

2. The London Mathematical Society having honoured our Society with exchange relations, the Proceedings of the London Mathematical Society in parts, as they are issued, will be available for issue at the Library, commencing with part I of Vol. XI (new series). Also, the Edinburgh Mathematical Society have similarly honoured us, and the Quarterly Notes published by them will also be available for issue.

3. The following books have been received from the Publishers for review in our Journal :—

(1) *Essai de Geometrie Analytique Modulairie a Deux Dimensions,*— by G. Arnoux, (Gauthier-Villars, Paris) 1911, 6 *fr.*

(2) *An Introduction to Geometry,*—by E. O. Taylor, Clarendon Press, Oxford, 1912, 1/6.

(3) *The Calculus for Beginners,*—by W. M. Baker, G. Bell & Sons, London, 1912, 3*s*/—.

(4) *Geometry for Schools,* Vols. I to IV,—by W. G. Borchardt and A. D. Perrott, G. Bell & Sons, London, 1912, 3/6.

POONA,
31st July, 1912.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theory of Moving Axes.

By A. C. L. Wilkinson, M.A., F.R.A.S.

(*Continued from page 55.*)

[The following sections deal with the general theory of curves on surfaces. The subject is developed entirely from first principles. References are made to Eisenhart: *A Treatise on the Differential Geometry of Curves and Surfaces.*]

§ 41. We may suppose any surface given by

$$x=\phi(u,v),\ y=\psi(u,v),\ z=\chi(u,v)$$

and any point on the surface may be denoted by the corresponding parameters (u,v).

The arc joining two points (u,v) and $(u+du,\ v+dv)$ is given by

$$ds^2=\mathrm{E}du^2+2\mathrm{F}dudv+\mathrm{G}dv^2 \quad \dots \quad \dots \quad \dots \quad (1)$$

where

$$\mathrm{E}=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2=\sum\left(\frac{\partial x}{\partial u}\right)^2$$

$$\mathrm{F}=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}=\sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}$$

$$\mathrm{G}=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=\sum\left(\frac{\partial x}{\partial v}\right)^2.$$

Equation (1) is called the first fundamental quadratic form.

The second fundamental quadratic form is defined by

$$\mathrm{D}\,du^2+2\,\mathrm{D}'\,du\,dv+\mathrm{D}''\,dv^2$$

where

$$\mathrm{D}=\frac{1}{\mathrm{H}}\sum\frac{\partial(y,z)}{\partial(u,v)}\frac{\partial^2 x}{\partial u^2}$$

$$\mathrm{D}'=\frac{1}{\mathrm{H}}\sum\frac{\partial(y,z)}{\partial(u,v)}\frac{\partial^2 x}{\partial u\partial v}$$

$$\mathrm{D}''=\frac{1}{\mathrm{H}}\sum\frac{\partial(y,z)}{\partial(u,v)}\frac{\partial^2 x}{\partial v^2}$$

and

$$\mathrm{H}^2=\mathrm{E}\mathrm{G}-\mathrm{F}^2=\sum\left\{\frac{\partial(y,z)}{\partial(u,v)}\right\}^2$$

It is easily seen that the square of the distance of the point $(u+du,\ v+dv)$ from the tangent plane at $(u,\ v)$ is $\mathrm{D}\,du^2+2\mathrm{D}'dudv+\mathrm{D}''dv^2$.

When the surface is real and likewise the parameters u, v, the functions E, F, G are all real and E, G, H^2 are essentially positive. We shall denote by $\sqrt{\mathrm{E}}$, $\sqrt{\mathrm{G}}$, H the positive square roots of these quantities.

There are three relations between the six quantities E, F, G, D, D′, D″ as we shall see later.

If we are given only the linear element or the functions E, F, G the surface is not completely determined, but we have a family of applicable surfaces or surfaces that are deformable into each other.

If in addition we are given D, D′, D″ subject to the necessary conditions, the surface is uniquely determined, except as regards its position in space. (Eisenhart § 65.)

The direction cosines of the tangents to $v=$constant, $u=$constant and the normal to the surface are

$$\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u},\ \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u},\ \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u}$$

$$\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v},\ \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v},\ \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v}$$

$$\frac{1}{H}\frac{\partial(y,z)}{\partial(u,v)},\ \frac{1}{H}\frac{\partial(z,x)}{\partial(u,v)}\ \frac{1}{H}\frac{\partial(x,y)}{\partial(u,v)}.$$

The angle ω between the parametric lines at the point (u, v) is given by

$$\cos\omega=\sum\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\cdot\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}=\frac{F}{\sqrt{EG}}$$

$$\sin\omega=\frac{H}{\sqrt{EG}}.$$

The following relations are easily obtained by differentiation:

$$\sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u^2}=\frac{1}{2}\frac{\partial E}{\partial u},\qquad \sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\partial v}=\frac{1}{2}\frac{\partial E}{\partial v}$$

$$\sum\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u\partial v}=\frac{1}{2}\frac{\partial G}{\partial u},\qquad \sum\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial v^2}=\frac{1}{2}\frac{\partial G}{\partial v}$$

$$\sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial v^2}=\frac{\partial F}{\partial v}-\frac{1}{2}\frac{\partial G}{\partial u},\qquad \sum\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2}=\frac{\partial F}{\partial u}-\frac{1}{2}\frac{\partial E}{\partial v}.$$

§ 42. We may first consider a special case, namely, when the parametric lines are orthogonal.

The element of the surface is now

$$ds^2=E\,du^2+G\,dv^2$$

for $F=o$ is the condition of orthogonality.

Take for moving axes, forming a right handed system, (1) the tangent line to $v=$constant: (2) the tangent line to $u=$constant: (3) the normal to the surface. We suppose the origin to move along a curve on the surface given by a functional relation between u and v; thus the motion of the origin is determined at each instant by the ratio $du : dv$.

The direction cosines of the moving axes are

$$\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u},\ \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u},\ \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u}$$

$$\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v},\ \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v},\ \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v}$$

$$\frac{1}{\sqrt{EG}}\frac{\partial(y,z)}{\partial(u,v)},\ \frac{1}{\sqrt{EG}}\frac{\partial(z,x)}{\partial(u,v)},\ \frac{1}{\sqrt{EG}}\frac{\partial(x,y)}{\partial(u,v)}$$

where $\sqrt{E}$, $\sqrt{G}$, $\sqrt{EG}$ are all positive real quantities for real values in a certain domain of u, v and the determinant of the nine direction cosines is $+1$.

To determine the infinitesimal rotations.

Relatively to the moving axes, the straight line

$$\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u},\ \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v},\ \frac{1}{\sqrt{EG}}\frac{\partial(y,z)}{\partial(u,v)}$$

and other similar straight lines are fixed in space. Whence by § 3, I. M. J., Vol. III, p. 95.

$$d\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right)-\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}\partial_3+\frac{1}{\sqrt{EG}}\frac{\partial(y,z)}{\partial(u,v)}\partial_2=0$$

$$d\left(\frac{1}{\sqrt{E}}\frac{\partial y}{\partial u}\right)-\frac{1}{\sqrt{G}}\frac{\partial y}{\partial v}\partial_3+\frac{1}{\sqrt{EG}}\frac{\partial(z,x)}{\partial(u,v)}\partial_2=0$$

$$d\left(\frac{1}{\sqrt{E}}\frac{\partial z}{\partial u}\right)-\frac{1}{\sqrt{G}}\frac{\partial z}{\partial v}\partial_3+\frac{1}{\sqrt{EG}}\frac{\partial(x,y)}{\partial(u,v)}\partial_2=0.$$

To solve these for ∂_2, ∂_3 multiply by $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$ and add : we obtain

$$\sqrt{G}\,\partial_3=\sum d\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right)\cdot\frac{\partial x}{\partial v}$$

$$=\sum\frac{1}{\sqrt{E}}\left(\frac{\partial^2 x}{\partial u^2}du+\frac{\partial^2 x}{\partial u\partial v}dv\right)\frac{\partial x}{\partial v},\ \text{since } F=0,$$

$$=\frac{1}{2\sqrt{E}}\left\{-\frac{\partial E}{\partial v}du+\frac{\partial G}{\partial u}dv\right\}$$

$$\partial_3=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}du+\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}dv \quad \ldots \quad \ldots \quad (1)$$

Again, multiply by $\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}$ and add : we obtain,

$$\sqrt{EG}\,\partial_2=-\frac{1}{\sqrt{E}}\sum\frac{\partial^2 x}{\partial u^2}\frac{\partial(y,z)}{\partial(u,v)}du-\frac{1}{\sqrt{E}}\sum\frac{\partial^2 x}{\partial u\,\partial v}\frac{\partial(y,z)}{\partial(u,v)}dv.$$

$$\therefore \quad \partial_2=-\frac{1}{\sqrt{E}}(D\,du+D'dv) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and similarly we find,

$$\partial_1=\frac{1}{\sqrt{G}}(D'du+D''dv) \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

For the displacement of the origin we have

$$\xi=\sqrt{E}\,du,\ \eta=\sqrt{G}\,dv,\ \zeta=0.$$

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§ 43. *The lines of curvature on the surface for which $F=0$.*

For a line of curvature consecutive normals intersect : the normal at $(\sqrt{E}\,du, \sqrt{G}\,dv, o)$ has for direction cosines $(\partial_2, -\partial_1, 1)$. If the normals

$$\frac{x}{0}=\frac{y}{0}=\frac{z}{1} \text{ and } \frac{x-\sqrt{E}\,du}{\partial_2}=\frac{y-\sqrt{G}\,dv}{-\partial_1}=\frac{z}{1}$$

intersect, we have

$$\sqrt{E}\,du\,\partial_1+\sqrt{G}\,dv\,\partial_2=0.$$

Hence the equation of the lines of curvature is :

$$(D'du+D''dv)\,Edu-(Ddu+D'dv)\,Gdv=0,$$

If the parametric lines $u=$constant $v=$constant, are taken as the lines of curvature, the above equation must reduce to $dudv=0$: hence :

The necessary and sufficient conditions that the parametric curves be the lines of curvature are $F=0$, $D'=0$.

§ 44. *The radii of curvature and torsion of the lines of curvature.*

Take the lines of curvature as parametric, so that $F=0$, $D'=0$.

Consider the curve $v=$constant: the direction cosines of the tangent are $(1, 0, 0)$ and of the consecutive tangent are $(1, \partial_3, -\partial_2)$, where ∂_3, ∂_2 are taken subject to $dv=0$.

Also $\Delta s=\sqrt{E}\,du$, whence $\Delta\varepsilon^2=\partial_3^2+\partial_2^2$ gives

$$\frac{1}{\rho^2}=\frac{1}{EG}\left(\frac{\partial\sqrt{E}}{\partial v}\right)^2+\frac{D^2}{E^2}.$$

Similarly for the curvature of the line $u=$constant, we find

$$\frac{1}{\rho^2}=\frac{1}{EG}\left(\frac{\partial\sqrt{G}}{\partial u}\right)^2+\frac{D''^2}{G^2}.$$

For the binormal to $v=$constant, it is perpendicular to the two consecutive tangents $(1, 0, 0)$ and $(1, \partial_3, -\partial_2)$ and is therefore :

$$(0, \cos\chi, \sin\chi), \text{ where } \cos\chi\,\partial_3-\sin\chi\,\partial_2=0.$$

Thus $$\Delta T^2=(-\cos\chi\partial_3+\sin\chi\partial_2)^2+(-\sin\chi d\chi-\sin\chi\partial_1)^2 +(\cos\chi d\chi+\cos\chi\partial_1)^2=d\chi^2,$$

since, for $v=$constant, $\partial_1=0$.

Thus $$\frac{1}{\sigma}=\pm\frac{1}{\sqrt{E}}\frac{\partial\chi}{\partial u}$$

where, $$\tan\chi=\frac{1}{D}\sqrt{\frac{E}{G}}\frac{\partial\sqrt{E}}{\partial v}.$$

The ambiguity can be removed by the use of Frenet's formulae of § 5. The axes fixed in space of § 5 may be taken to be the moving axes at the origin, and then

$$\left.\begin{matrix} l_1, m_1, n_1 \\ l_2, m_2, n_2 \\ l_3, m_3, n_3 \end{matrix}\right\} \text{are respectively} \left\{\begin{matrix} 1, & 0, & 0, \\ 0, & \sin\chi, & -\cos\chi, \\ 0, & \cos\chi, & \sin\chi. \end{matrix}\right.$$

We might take $(0, -\sin\chi, \cos\chi)$ and $(0, -\cos\chi, -\sin\chi)$ for the principal normal and binormal respectively ; the result will be the same.

Now $(l+dl_3, m+dm_3, n+dn_3)$ are, as just determined in the expression for dT^2, $(0, \cos\chi - \sin\chi d\chi, \sin\chi + \cos\chi d\chi)$, whence Frenet's formula $dn_3/ds = -n_2/\sigma$ gives :

$$\frac{\cos\chi d\chi}{ds} = \frac{\cos\chi}{\sigma}, \text{ or } \frac{1}{\sigma} = \frac{d\chi}{ds}.$$

Thus for the torsion of v=constant, we have

$$\frac{1}{\sigma} = \frac{1}{\sqrt{E}} \frac{\partial \chi}{\partial u}, \text{ where } \tan\chi = \frac{1}{D}\sqrt{\frac{E}{G}} \frac{\partial \sqrt{E}}{\partial v};$$

and similarly for the torsion of u=constant, we shall find

$$\frac{1}{\sigma} = -\frac{1}{\sqrt{G}} \frac{\partial \chi'}{\partial v} \text{ where } \tan\chi' = \frac{1}{D''}\sqrt{\frac{G}{E}} \frac{\partial \sqrt{G}}{\partial u}.$$

If the line of curvature, v=constant, is plane, we have χ a function of v only ; thus χ is constant along the line of curvature. In other words, the binormal or perpendicular to the plane of the line of curvature makes a constant angle with the principal normal to the surface :

If a line of curvature is plane, its plane cuts the surface at a constant angle and conversely.

Again, if a line of curvature is a geodesic, we must have $\chi=0$, whence it is plane.

§ 45. We may determine the radius of curvature of a curve cutting the lines v=constant, at an angle θ.

The moving axes are the tangents to v=constant, u=constant and the normal to the surface, we displace the origin along the curve making an angle θ with the line v=constant, whence, if ds is the arc of the curve,

$$\sqrt{E}\, du = ds \cos\theta, \quad \sqrt{G}\, dv = ds \sin\theta.$$

The direction cosines of the tangent are $(\cos\theta, \sin\theta, 0)$, and for the consecutive tangent, we have

$(\cos\theta - \sin\theta\, d\theta - \sin\theta\, \partial_3, \sin\theta + \cos\theta\, d\theta + \cos\theta\, \partial_3, -\cos\theta\, \partial_2 + \sin\theta\, \partial_1)$

$$\text{Thus } \frac{1}{\rho^2} = \left(\frac{d\theta}{ds} + \frac{\partial_3}{ds}\right)^2 + \left(\cos\theta \frac{\partial_2}{ds} - \sin\theta \frac{\partial_1}{ds}\right)^2$$

$$= \left\{\frac{d\theta}{ds} + \frac{1}{\sqrt{EG}}\left(\sin\theta \frac{\partial\sqrt{G}}{\partial v} - \cos\theta \frac{\partial\sqrt{E}}{\partial u}\right)\right\}^2$$

$$+ \left\{\frac{D}{E}\cos^2\theta + \frac{2D'}{\sqrt{EG}} \sin\theta\cos\theta + \frac{D''}{G}\sin^2\theta\right\}^2;$$

on substituting for $\partial_1, \partial_2, \partial_3$ their values from § 24.

The radius of torsion will be obtained by another method.

The condition for a geodesic.

The binormal for a geodesic must have direction cosines $(\cos\lambda, \sin\lambda, 0)$, if this is perpendicular to the two tangents obtained above, we have

$$d\theta+\partial_3=0.$$

§ 46. *Gauss's Theorem relating to geodesics.*

Take $v=$constant as a family of geodesic lines, and $u=$constant as their orthogonal trajectories. Since $\theta=0$, gives a geodesic, we have $\partial_3=0$, for displacements along $v=$constant ; thus $\frac{\partial\sqrt{E}}{\partial v}=0$, and E is a function of u only. Writing $\sqrt{E}du=dU$, the element of the surface becomes

$$ds^2=dU^2+Gdv^2.$$

Conversely, if the element of a surface has this form, the curves $v=$constant are geodesics.

Consider the arcs of any two geodesics intercepted between the curves $U=U_0$ and $U=U_1$, we have

$$S=\int_{U_0}^{U_1} dU=U_1-U_0$$

This is the important theorem due to Gauss :

If from any point on a curve on a surface geodesics of constant length are drawn perpendicular to the curve, the locus of their extremities is another curve orthogonal to the geodesics.

Some simple theorems are easily proved.

(1) Suppose on a surface two families of geodesics cut orthogonally. We have E a function of u only, G a function of v only and the surface can be represented by $ds^2=du^2+dv^2$, namely, the surface is applicable to a plane.

Thus the only surfaces on which two sets of geodesics cut orthogonally are developable surfaces and on them an infinite number of such orthogonal series can be drawn.

(2) If a line of curvature is a geodesic, it is plane.

Refer the surface to its lines of curvature, along the geodesic $v=$constant, we have $D'=0$, $\theta=0$ whence $\partial_3=0$.

The binormal is $(0, 1, 0)$ and the consecutive binormal is $(-\partial_3, 1, \partial_1)$ which reduces to $(0, 1, 0)$, thus the curve is plane.

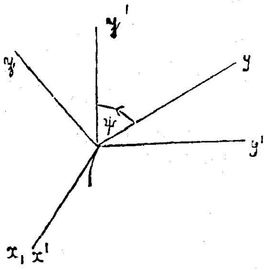
(3) If a geodesic line is a plane curve, it is a line of curvature.

Take a system of geodesics including the given one and their orthogonal trajectories.

The surface is $ds^2 = du^2 + G\, dv^2$.

The consecutive binormal to (0, 1, 0) is $(0, 1, \partial_1)$ and for a plane curve we must have $\partial_1 = 0$, for v = some constant, whence $D' = 0$ along the line. The line is therefore a line of curvature.

§ 47· Following Cesaro, '*Geometria Intrinseca* (pp. 152 to 158) a second system of moving axes can be introduced as follows :—



Consider any curve on a surface, take the tangent, principal normal and binormal as axes of x, y, z and the tangent to the curve, the perpendicular to this tangent in the tangent plane of the surface and the normal to the surface as axes of x', y', z'. Let ψ be the angle between the principal normal to the curve and the normal to the surface measured towards the binormal.

Write $n = \cos\psi/\rho$, $g = \sin\psi/\rho$: n, g are called respectively the *normal curvature* and the *geodesic curvature* of the curve on the surface. A third quantity t called the *geodesic torsion* may be introduced as follows : we write down the condition the curve may be a line of curvature, the moving trihedral is now that of § 4 formed by the tangent, principal normal and binormal to a curve. The direction cosines of the normal to the surface are $(0, \cos\psi, \sin\psi)$. The equations of the two consecutive normals to the surface along the curve we are considering are

$$\frac{x}{0} = \frac{y}{\cos\psi} = \frac{z}{\sin\psi}$$

and

$$\frac{x - ds}{-\cos\psi d\varepsilon} = \frac{y}{\cos\psi - \sin\psi d\psi - \sin\psi dT} = \frac{z}{\sin\psi + \cos\psi d\psi + \cos\psi dT}.$$

The condition they intersect is $d\psi + dT = 0$.

We define a quantity $t=\frac{d\psi}{ds}+\frac{1}{\sigma}$, and call it the *geodesic torsion* of the curve.

It may be noted that the quantities I have defined as the torsion and geodesic torsion have the opposite signs to the corresponding quantities in Eisenhart.

The asymptotic lines of the surface are given by $n=0$, the geodesic lines of the surface are given by $g=0$, and the lines of curvature of the surface are given by $t=0$.

We may note here Joachimstal's theorem :

If a line of intersection of two surfaces is a line of curvature on both, the surfaces intersect at a constant angle.

For
$$\frac{d(\psi_1-\psi_2)}{ds}=0.$$

Again, if a line of curvature is a geodesic, since ψ is everywhere zero, therefore also $1/\sigma$ is zero, and the line of curvature is plane.

If now (x,y,z), (x',y',z') are the same point referred to the two trihedrals, we have

$$x=x',\ y=y'\sin\psi+z'\cos\psi,\ z=-y'\cos\psi+z'\sin\psi$$
$$x'=x,\ y'=y\sin\psi-z\cos\psi,\ z'=y\cos\psi+z\sin\psi.$$

A point $(x+dx,\ y+dy,\ z+dz)$ or $(x'+dx',\ y'+dy',\ z'+dz')$ referred to the two new trihedrals, will have as its co-ordinates referred to the old trihedrals $(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ and $(x'+\Delta x',\ y'+\Delta y',\ z'+\Delta z')$.

For Δx, Δy, Δz we have from §§ 2 and 4.

$$\frac{\Delta x}{\Delta s}=\frac{dx}{ds}+1-\frac{y}{\rho},\ \frac{\Delta y}{\Delta s}=\frac{dy}{ds}-\frac{z}{\sigma}+\frac{x}{\rho},\ \frac{\Delta z}{\Delta s}=\frac{dz}{ds}+\frac{y}{\sigma},$$

Also $\Delta x'=\Delta x$, $\Delta y'=\Delta y\sin\psi-\Delta z\cos\psi$, $\Delta z'=\Delta y\cos\psi+\Delta z\sin\psi$.

Remembering for the relations between dx, dy, dz and dx', dy', dz', that the angle $\psi+d\psi$ occurs instead of ψ, we have

$$dx=dx',$$
$$dy=dy'\sin\psi+dz'\cos\psi+(y'\cos\psi-z'\sin\psi)\,d\psi,$$
$$dz=-dy'\cos\psi+dz'\sin\psi+(y'\sin\psi+z'\cos\psi)\,d\psi.$$

whence
$$\frac{\Delta x'}{\Delta s}=\frac{dx}{ds}+1-\frac{y}{\rho}=\frac{dx'}{ds}+1-y'g-z'n.$$
$$\frac{\Delta y'}{\Delta s}=\sin\psi\left(\frac{dy}{ds}-\frac{z}{\sigma}+\frac{x}{\rho}\right)-\cos\psi\left(\frac{dz}{ds}+\frac{y}{\sigma}\right)$$
$$=\frac{dy'}{ds}-z't+x'g,$$
$$\frac{\Delta z'}{\Delta s}=\frac{dz'}{ds}+y't+x'n.$$

Thus, when we consider any curve on a surface, and take as moving axes the tangent to the curve, the perpendicular to the tangent in the tangent plane to the surface and the normal to the surface, forming a right-handed system, the absolute displacements in space of a point (x, y, z) are given by the formulæ :

$$\frac{\Delta x}{\Delta s}=\frac{dx}{ds}-yg-zn+1,$$

$$\frac{\Delta y}{\Delta s}=\frac{dy}{ds}-zt+xg,$$

$$\frac{\Delta z}{\Delta s}=\frac{dz}{ds}+yt+xn,$$

where n, g, t are, as defined above, the normal curvature, geodesic curvature and geodesic torsion.

Similarly for the direction cosines of a variable line (l, m, n) we have only to suppress the unity in the first equation.

We may note here that n, t deqend solely on the tangent to the curve and are the same for all curves on the surface having a common tangent. For the consecutive normal to the surface to $(0, 0, 1)$ has for its direction cosines $(-nds, -tds, 1)$ and is the same for all curves with the same tangent. That n is the same for all curves with the same tangent is of course only Meunier's Theorem, that t is the same is a theorem due to Bonnet.

§ 48. Consider now a surface in the form

$$ds^2=\mathrm{E}du^2+\mathrm{G}dv^2.$$

Taking the tangent to $v=$constant, its perpendicular in the tangent plane to the surface and the normal to the surface as axes, we have, for the displacement of any point (xyz), the equations

$$\frac{\Delta x}{\Delta s}=\frac{dx}{ds}-yg-zn+1=\frac{dx}{ds}-y\frac{\partial_3}{ds}+z\frac{\partial_2}{ds}+1$$

$$\frac{\Delta y}{\Delta s}=\frac{dy}{ds}-zt+xg=\frac{dy}{ds}-z\frac{\partial_1}{ds}+x\frac{\partial_3}{ds}$$

$$\frac{\Delta z}{\Delta s}=\frac{dz}{ds}+yt+xn=\frac{dz}{ds}-x\frac{\partial_2}{ds}+y\frac{\partial_1}{ds}$$

where $\partial_1, \partial_2, \partial_3$ are taken on the supposition $v=$constant and $ds=\sqrt{\mathrm{E}}\,du$. Substitute for $\partial_1, \partial_2, \partial_3$ their values for $v=$constant, from §42 and solve ; we find :

$$n=\frac{\mathrm{D}}{\mathrm{E}},\ g=-\frac{1}{\sqrt{\mathrm{G}}}\frac{\partial \log \sqrt{\mathrm{E}}}{\partial v},\ t=\frac{\mathrm{D}'}{\sqrt{\mathrm{EG}}}.$$

To obtain the similar expressions for u=constant, we must write in the first set of equations y for x and $-x$ for y and with $ds=\sqrt{G}\, dv$ we have

$$\frac{dx}{ds}+zt'-yg'=\frac{dx}{ds}-y\frac{\partial_3}{ds}+z\frac{\partial_2}{ds}$$

$$\frac{dy}{ds}+xg'-zn'+1=\frac{dy}{ds}-z\frac{\partial_1}{ds}+x\frac{\partial_3}{ds}+1$$

$$\frac{dz}{ds}-xt'+yn'=\frac{dz}{ds}-x\frac{\partial_2}{ds}+y\frac{\partial_1}{ds}$$

where ∂_1, ∂_2, ∂_3 are taken on the supposition u=constant.

Hence $n'=\frac{D''}{G}$, $g'=\frac{1}{\sqrt{E}}\frac{\partial \log \sqrt{G}}{\partial u}$, $t'=-\frac{D'}{\sqrt{EG}}$.

We note the theorem: for any two perpendicular directions on a surface $t+t'=0$.

These results are a particular case of the next section.

§. 49. To consider a curve making an angle θ with the line v=constant, we must turn the axes in the x, y plane through an angle θ. The formulæ to be transformed are

$$\Delta x=dx+\xi-y\partial_3+z\partial_2$$
$$\Delta y=dy+\eta-z\partial_1+x\partial_3$$
$$\Delta z=dz-x\partial_2+y\partial_1,$$

where $\xi=\sqrt{E}\, du$, $\eta=\sqrt{G}\, dv$.

For a direction θ, we have $\sqrt{E}\, du=ds\cos\theta$, $\sqrt{G}\, dv=ds\sin\theta$, and the transformation formulæ:

$$x'=x\cos\theta+y\sin\theta,\quad y'=y\cos\theta-x\sin\theta,\quad z'=z,$$
$$x=x'\cos\theta-y'\sin\theta,\quad y=x'\sin\theta+y'\cos\theta,\quad z=z'.$$

For the absolute displacements in space $\Delta x'$, $\Delta y'$, $\Delta z'$, we have

$$\begin{aligned}\Delta x'&=\Delta x\cos\theta-\Delta y\sin\theta\\&=\xi\cos\theta+\eta\sin\theta+dx\cos\theta+dy\sin\theta\\&\qquad+(x\sin\theta-y\cos\theta)\partial_3+z\cos\theta\partial_1-z\sin\theta\partial_2\end{aligned}$$

which, by use of $dx'=dx\cos\theta+dy\sin\theta-x\sin\theta d\theta+y\cos\theta d\theta$, gives

$$\Delta x'=ds+dx'-y'(d\theta+\partial_3)+z'(\cos\theta\partial_2-\sin\theta\partial_1)$$

Similarly

$$\Delta y'=dy'+x'(d\theta+\partial_3)-z'(\sin\theta\partial_2+\cos\theta\partial_1)$$
$$\Delta z'=dz'-x'(\cos\theta\partial_2-\sin\theta\partial_1)+y'(\sin\theta\partial_2+\cos\theta\partial_1).$$

Comparing these with

$$\Delta x'=ds+dx'-y'Gds-z'Nds$$
$$\Delta y'=dy'-z'Tds+x'Gds$$
$$\Delta z'=dz'+y'Tds+x'Nds$$

where N, G, T are the normal curvature, geodesic curvature and geodesic torsion of the curve making an angle θ with v=constant, we obtain:

$$\begin{aligned} N ds &= \sin\theta\,\partial_1 - \cos\theta\,\partial_3 \\ G ds &= d\theta + \partial_3 \\ T ds &= \sin\theta\,\partial_2 + \cos\theta\,\partial_1. \end{aligned}$$

Substitute for ∂_1, ∂_2, ∂_3 their values from § 42, and replace $\frac{\sqrt{E}du}{ds}$ by $\cos\theta$ and $\frac{\sqrt{G}du}{ds}$ by $\sin\theta$, and we have:

$$N = \frac{D}{E}\cos^2\theta + \frac{2D'}{\sqrt{EG}}\sin\theta\cos\theta + \frac{D''}{G}\sin^2\theta$$

$$T = \frac{D'}{\sqrt{EG}}(\cos^2\theta - \sin^2\theta) + \left(\frac{D''}{G} - \frac{D}{E}\right)\sin\theta\cos\theta$$

$$G = \frac{d\theta}{ds} - \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v}\cos\theta + \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u}\sin\theta.$$

The first of these equations is simply Euler's Theorem. Writing $\theta=0$, $\theta=\frac{\pi}{2}$, we verify the results of the previous section.

Forming N^2+G^2 we obtain the result of § 45 for the curvature of the curve making an angle θ with v=constant.

For the radius of torsion, we have $T = d\psi/ds + 1/\sigma$, where $\tan\psi = G/N$. Thus

$$\frac{1}{\sigma} = T - \frac{N\frac{dG}{ds} - G\frac{dN}{ds}}{N^2+G^2}$$

where $$\frac{d}{ds} = \frac{\cos\theta}{\sqrt{E}}\frac{\partial}{\partial u} + \frac{\sin\theta}{\sqrt{G}}\frac{\partial}{\partial v}.$$

Putting $\theta=0$, $\theta=\frac{\pi}{2}$, we may verify the formulae for the radii of torsion of the lines of curvature of § 44, where $D'=0$.

The condition for a geodesic is $G=0$, or $d\theta+\partial_3=0$, also the lines of curvature are given by $\sin\theta\partial_2+\cos\theta\partial_3=0$, and the asymptotic lines by $\sin\theta\partial_1-\cos\theta\partial_3=0$.

§. 50. Consider now the surface given in its most general form:

$$ds^2 = E du^2 + 2F du\, dv + G dv^2.$$

We have from § 41

$$\cos\omega = \frac{F}{\sqrt{EG}},\quad \sin\omega = \frac{H}{\sqrt{EG}}$$

where H, $\sqrt{E}$, $\sqrt{G}$ are positive quantities.

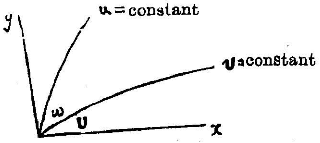
Differentiating

$$-\sin\omega d\omega = \frac{1}{\sqrt{EG}}dF + Fd\frac{1}{\sqrt{EG}},$$

or

$$d\omega = -\frac{1}{H}dF + \frac{F}{\sqrt{EG}}\frac{d\sqrt{EG}}{H}$$

Take axes x, y in the tangent plane at (u, v) to the surface, the axis of x making an angle U with the curve v=constant.



We suppose the origin displaced to the point $(u+du, v+dv)$, whence by projection

$$\xi = \sqrt{E}du \cos U + \sqrt{G}dv \cos(U+\omega),$$

$$\eta = \sqrt{E}du \sin U + \sqrt{G}dv \sin(U+\omega),$$

$$\zeta = 0.$$

The direction cosines, referred to the fixed axes, of the tangents to v=constant, u=constant are:

$$\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}, \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u}, \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u} \text{ and } \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}, \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v}, \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v}.$$

The direction cosines of the axis of x will be

$$\frac{\lambda}{\sqrt{E}}\frac{\partial x}{\partial u} + \frac{\mu}{\sqrt{G}}\frac{\partial x}{\partial v}, \ldots, \ldots$$

where

$$\cos U = \lambda + \mu \cos\omega, \ \cos(U+\omega) = \lambda \cos\omega + \mu$$

Solving for λ, μ, the direction cosines of the axis of x are:

$$\frac{\sin(U+\omega)}{\sin\omega}\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u} - \frac{\sin U}{\sin\omega}\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}, \ldots, \ldots$$

Similarly, the direction cosines of the axis of y are:

$$-\frac{\cos(U+\omega)}{\sin\omega}\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u} + \frac{\cos U}{\sin\omega}\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}, \ldots, \ldots$$

and the normal to the surface forming with these a righthanded system is

$$\frac{1}{H}\frac{\partial(y, z)}{\partial(u, v)}, \ldots, \ldots$$

The determination of ∂_1, ∂_2, ∂_3 proceeds exactly as in § 42.

We have

$$d\left\{\frac{1}{H}\frac{\partial(y,z)}{\partial(u,v)}\right\}-\partial_2\left\{\frac{\sin(U+\omega)}{\sin\omega}\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}-\frac{\sin U}{\sin\omega}\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}\right\}$$

$$+\partial_1\left\{-\frac{\cos(U+\omega)}{\sin\omega}\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}+\frac{\cos U}{\sin\omega}\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}\right\}=0,$$

and two similar equations.

Multiply these by $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$ and add, and also by $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$ and add : noticing that

$$\sum d\frac{\partial(y,z)}{\partial(u,v)}.\frac{\partial x}{\partial u}=-\sum\frac{\partial(y,z)}{\partial(u,v)}.\ d\ \frac{\partial x}{\partial u}$$

we find

$$-(D\,du+D'dv)-\partial_2\left\{\sqrt{E}\frac{\sin(U+\omega)}{\sin\omega}-\frac{F}{\sqrt{G}}\frac{\sin U}{\sin\omega}\right\}$$

$$+\partial\left\{-\frac{\cos(U+\omega)}{\sin\omega}\sqrt{E}+\frac{F}{\sqrt{G}}\frac{\cos U}{\sin\omega}\right\}=0,$$

$$-(D'\,du+D''\,dv)-\partial_2\left\{\frac{F}{\sqrt{E}}\frac{\sin(U+\omega)}{\sin\omega}-\sqrt{G}\frac{\sin U}{\sin\omega}\right\}$$

$$+\partial_1\left\{-\frac{F}{\sqrt{E}}\frac{\cos(U+\omega)}{\sin\omega}+\sqrt{G}\frac{\cos U}{\sin\omega}\right\}=0.$$

Reducing these by means of $F=\sqrt{EG}\cos\omega$, they become

$$-(D\,du+D'\,dv)-\partial_2\sqrt{E}\cos U+\partial_1\sqrt{E}\sin U=0,$$

$$-(D'\,du+D''\,dv)-\partial_2\sqrt{G}\cos(U+\omega)+\partial_1\sqrt{G}\sin(U+\omega)=0;$$

whence solving

$$\partial_1=-\frac{D\,du+D'\,dv}{\sqrt{E}}\frac{\cos(U+\omega)}{\sin\omega}+\frac{D'\,du+D''\,dv}{\sqrt{G}}\frac{\cos U}{\sin\omega},$$

$$\partial_2=-\frac{D\,du+D'\,dv}{\sqrt{E}}\frac{\sin(U+\omega)}{\sin\omega}+\frac{D'\,du+D''\,dv}{\sqrt{G}}\frac{\sin U}{\sin\omega}.$$

To determine ∂_3, we have

$$d\left\{\frac{\sin(U+\omega)}{\sin\omega}\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}-\frac{\sin U}{\sin\omega}\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}\right\}$$

$$-\partial_3\left\{-\frac{\cos(U+\omega)}{\sin\omega}\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}+\frac{\cos U}{\sin\omega}\frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}\right\}$$

$$+\partial_2\frac{1}{H}\frac{\partial(y,z)}{\partial(u,v)}=0,$$

and two similar equations.

Multiply by $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$ and add, the co-efficient of ∂_3 reduces to $\sqrt{E}\sin U$, and writing the other part

$$\Sigma\, d \left[\sin U \left\{ \frac{F}{H\sqrt{E}} \frac{\partial x}{\partial u} - \frac{\sqrt{E}}{H} \frac{\partial x}{\partial v} \right\} + \cos U \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u} \right] \cdot \frac{\partial x}{\partial u}$$

$$= \sin U\, \Sigma\, d \left(\frac{F}{H\sqrt{E}} \frac{\partial x}{\partial u} - \frac{\sqrt{E}}{H} \frac{\partial x}{\partial v} \right) \cdot \frac{\partial x}{\partial u} - \sqrt{E} \sin U dU,$$

since $\sum \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u} d \left(\frac{1}{\sqrt{E}} \frac{\partial x}{\partial u} \right) = \frac{1}{2} \Sigma\, d \left\{ \frac{1}{E} \left(\frac{\partial x}{\partial u} \right)^2 \right\} = 0,$

we find for ∂_3 the expression :

$$\partial_3 = -dU + \frac{1}{H} d \frac{F}{H} - \frac{1}{H} \left\{ \frac{1}{2} \frac{\partial E}{\partial v} du + \left(\frac{\partial F}{\partial v} - \frac{1}{2} \frac{\partial G}{\partial u} \right) dv \right\} - \frac{F}{H} d \frac{1}{H} - \frac{F}{H\sqrt{E}} d\sqrt{E}$$

$$= -dU + \frac{1}{H} \left\{ dF - \frac{1}{2} \frac{\partial E}{\partial v} du - \left(\frac{\partial F}{\partial v} - \frac{1}{2} \frac{\partial G}{\partial u} \right) dv - \frac{1}{2} \frac{F}{E} dE \right\}.$$

We may render this symmetrical by means of

$$d\omega = -\frac{1}{H} dF + \frac{F}{\sqrt{EG}} \frac{d\sqrt{EG}}{H},$$

and it becomes

$$\partial_3 = -\frac{1}{2} dU - \frac{1}{2} d(U+\omega) + \frac{1}{2H} \left\{ \frac{\partial F}{\partial u} du - \frac{\partial F}{\partial v} dv + \frac{\partial G}{\partial u} dv - \frac{\partial E}{\partial v} du + \frac{F}{2} d \log \frac{G}{E} \right\}.$$

Thus, if the moving axes make angles U, V with the parametric lines v = constant, u = constant, we have for a displacement corresponding to changes du, dv in u, v respectively :

$$\partial_1 = -\frac{1}{\sqrt{E}} \frac{\cos V}{\sin \omega} (D du + D' dv) + \frac{1}{\sqrt{G}} \frac{\cos U}{\sin \omega} (D' du + D'' dv)$$

$$\partial_2 = -\frac{1}{\sqrt{E}} \frac{\sin V}{\sin \omega} (D du + D' dv) + \frac{1}{\sqrt{G}} \frac{\sin U}{\sin \omega} (D' du + D'' dv)$$

$$\partial_3 = -\frac{1}{2} d(U+V) + \frac{1}{2H} \left\{ \frac{\partial F}{\partial u} du - \frac{\partial F}{\partial v} dv + \frac{\partial G}{\partial u} dv - \frac{\partial E}{\partial v} du + \frac{F}{2} d \log \frac{G}{E} \right\}.$$

$\xi = \sqrt{E} du \cos U + \sqrt{G} dv \cos V$, $\eta = \sqrt{E} du \sin U + \sqrt{G} dv \sin V$,
$\zeta = 0$, $V - U = \omega$.

It remains to identify these equations with those given in Darboux, Vol. II, p. 348.

§ 51. Suppose we move the origin along the curve making an angle U with v = constant, let N, J, T be the normal curvature geodesic curvature and geodesic torsion of this curve.

We have $\xi=ds$, $\eta=0$, $\zeta=0$; and for the absolute displacement of a point (x, y, z), we have

$$\begin{aligned}\Delta x&=dx+ds-y\partial_3+z\partial_2=dx+ds-yJds-zNds\\ \Delta y&=dy-z\partial_1+x\partial_3=dy-zTds+xJds\\ \Delta z&=dz-x\partial_2+y\partial_1=dz+yTds+xNds\end{aligned}$$

where the second system of equations are those of Cesaro obtained in § 47.

Thus we have $Tds=\partial_1$, $Nds=-\partial_2$, $Jds=\partial_3$.

If n, g, t are these quantities for the curve $v=$constant, we put $U=0$, $V=\omega$, $v=$constant, $ds=\sqrt{E}\,du$, and find

$$t=\frac{ED'-FD}{HE}$$

$$n=\frac{D}{E}$$

$$g=\frac{1}{H\sqrt{E}}\left(\frac{\partial F}{\partial u}-\tfrac{1}{2}\frac{F}{E}\frac{\partial E}{\partial u}-\tfrac{1}{2}\frac{\partial E}{\partial v}\right)$$

Similarly, if n', g', t' are the same quantities for $u=$constant, for which $U=-\omega$, $V=0$, $ds=\sqrt{G}\,dv$, we have

$$t'=\frac{FD''-GD'}{HG}$$

$$n'=\frac{D''}{G}$$

$$g'=\frac{1}{H\sqrt{G}}\left\{-\frac{\partial F}{\partial v}+\tfrac{1}{2}\frac{F}{G}\frac{\partial G}{\partial v}+\tfrac{1}{2}\frac{\partial G}{\partial u}\right\}.$$

We proceed to the determination of T, N, J for the curve, whose element of variation is given by $du : dv$.

The angle U is determined by

$$ds=\sqrt{E}\,du\cos U+\sqrt{G}\,dv\cos(U+\omega),$$
$$0=\sqrt{E}\,du\sin U+\sqrt{G}\,dv\sin(U+\omega);$$

whence we obtain

$$ds\sin(U+\omega)=\sqrt{E}\,du\sin\omega,\quad ds\cos U=\sqrt{E}\,du+\sqrt{G}dv\cos\omega$$
$$ds\sin U=-\sqrt{G}\,dv\sin\omega,\quad ds\cos(U+\omega)=\sqrt{E}\,du\cos\omega+\sqrt{G}\,dv.$$

Substitute for $\sin\omega$, $\cos\omega$ their values $\frac{H}{\sqrt{EG}}, \frac{F}{\sqrt{EG}}$ and substitute in the expressions $\frac{\partial_1}{ds}, \frac{\partial_2}{ds}, \frac{\partial_3}{ds}$ and we obtain

$$T=-\frac{1}{H}\left(D\frac{du}{ds}+D'\frac{dv}{ds}\right)\left(F\frac{du}{ds}+G\frac{dv}{ds}\right)$$
$$+\frac{1}{H}\left(D'\frac{du}{ds}+D''\frac{dv}{ds}\right)\left(E\frac{du}{ds}+F\frac{dv}{ds}\right),$$

$$N=\left(D\frac{du}{ds}+D'\frac{dv}{ds}\right)\frac{du}{ds}+\left(D'\frac{du}{ds}+D''\frac{dv}{ds}\right)\frac{dv}{ds},$$

$$J=-\frac{dU}{ds}+\frac{1}{H}\left\{\frac{\partial F}{\partial u}\frac{du}{ds}-\frac{1}{2}\frac{\partial E}{\partial v}\frac{du}{ds}+\frac{1}{2}\frac{\partial G}{\partial u}\frac{dv}{ds}-\frac{1}{2}\frac{F}{E}\left(\frac{\partial E}{\partial u}\frac{du}{ds}+\frac{\partial E}{\partial v}\frac{dv}{ds}\right)\right\}$$

To identify J with the expression given by the second differential parameter, due to Bonnet, we consider,

$$J=\frac{1}{H}\left[\frac{\partial}{\partial u}\left\{\sqrt{G}\cos(U+\omega)\right\}-\frac{\partial}{\partial v}\left\{\sqrt{E}\cos U\right\}\right].$$

The two expressions for J are seen to be identical on expanding the second expression; for, we have

$$\frac{1}{H}\left\{-\sqrt{G}\sin(U+\omega)\frac{\partial U}{\partial u}+\sqrt{E}\sin U\frac{\partial U}{\partial v}\right\}+\frac{\cos(U+\omega)}{H}\frac{\partial}{\partial u}\sqrt{G}$$

$$-\frac{\cos U}{H}\frac{\partial}{\partial v}\sqrt{E}-\frac{\sqrt{G}}{H}\sin(U+\omega)\frac{\partial\omega}{\partial u}$$

$$=-\left(\frac{\partial U}{\partial u}\frac{du}{ds}+\frac{\partial U}{\partial v}\frac{dv}{ds}\right)+\frac{1}{H}\left[\left(\frac{F}{\sqrt{G}}\frac{du}{ds}+\sqrt{G}\frac{dv}{ds}\right)\frac{\partial\sqrt{G}}{\partial u}-\right.$$

$$\left.\left(\sqrt{E}\frac{du}{ds}+\frac{F}{\sqrt{E}}\frac{dv}{ds}\right)\frac{\partial\sqrt{E}}{dv}\right]$$

$$-\frac{du}{ds}\left\{-\frac{1}{H}\frac{\partial F}{\partial u}+\frac{F}{H}\frac{\partial\log\sqrt{EG}}{\partial u}\right\}$$

and a comparison of the two values of J will show that they are identical.

We thus have

$$J=\frac{1}{H}\left\{\frac{\partial}{\partial u}\left(F\frac{du}{ds}+G\frac{dv}{ds}\right)-\frac{\partial}{\partial v}\left(F\frac{dv}{ds}+E\frac{du}{ds}\right)\right\}$$

which is Bonnet's formulæ.

Since T depends only on du/ds, dv/ds, it is the same for all curves at the point which have the same tangent and by considering the geodesic at the point in the given direction, since by definition $t=d\psi/ds+1/\sigma$ and ψ is everywhere zero for geodesic, we have the important theorem :

The geodesic torsion of a curve on a surface at a point is the torsion of the geodesic line touching the curve at the given point.

The radii of curvature and torsion of this curve are given by

$$\frac{1}{\rho^2}=J^2+N^2,\quad \frac{1}{\sigma}=T-\frac{d}{ds}\tan^{-1}\frac{J}{N}.$$

§. 52. *All surfaces on which two sets of geodesics cut at a constant angle are developables.*

The condition for a geodesic is $J=0$. If $u=$constant, are geodesics, we have

$$\frac{1}{2}\frac{F}{G}\frac{\partial G}{\partial v}-\frac{\partial F}{\partial v}+\frac{1}{2}\frac{\partial G}{\partial u}=0,$$

or
$$\frac{\partial}{\partial v}\left(\frac{F}{\sqrt{G}}\right)-\frac{\partial}{\partial u}\sqrt{G}=0.$$

If also $v=$constant, are geodesics, we have

$$\frac{\partial}{\partial u}\left(\frac{F}{\sqrt{E}}\right)-\frac{\partial}{\partial v}\sqrt{E}=0,$$

also $\quad F=\sqrt{EG}\cos\alpha$, where α is constant.

$$\therefore \quad \cos\alpha\frac{\partial}{\partial u}\sqrt{G}=\frac{\partial}{\partial v}\sqrt{E},$$

and
$$\cos\alpha\frac{\partial}{\partial v}\sqrt{E}=\frac{\partial}{\partial u}\sqrt{G}.$$

Excluding $\cos^2\alpha=1$, when the geodesic lines coincide, we have

$$\frac{\partial}{\partial u}\sqrt{G}=0=\frac{\partial}{\partial v}\sqrt{E}.$$

Thus E is a function of u only, and G a function of v only ; and the surface is of the form $ds^2=du^2+dv^2$, which is applicable to a plane.

§ 53. The following method of defining geodesic curvature may be easily verified :

If A is a point on a curve on a surface and B a consecutive point the geodesic angle of contingence is the angle between the tangent at A to the curve and the projection of the tangent at B on the tangent plane to the surface at A. The geodesic curvature is the ratio of this angle to the arc AB.

Taking the tangent at A to the curve as

$$\frac{x}{1}=\frac{y}{o}=\frac{z}{o},$$

The consecutive tangent is

$$\frac{x-ds}{1}=\frac{y}{\partial_3}=\frac{z}{-\partial_2}.$$

The projection of this line on $z=o$ is $x\,\partial_3-y=o$, and the angle between this straight line and $y=o$, in the plane $z=o$, is ∂_3, which proves the theorem.

§ 54. In the expression for ∂_1, *viz.*,

$$\partial_1=-\frac{1}{\sqrt{E}}\frac{\cos(U+\omega)}{\sin\omega}(D\,du+D'dv)+\frac{1}{\sqrt{G}}\frac{\cos U}{\sin\omega}(D'du+D''dv)$$

substitute for sin ω, cos ω their values and collecting the terms in du, dv; we get :

$$\partial_1=\left(\cos U.\frac{D'E-DF}{EH}+\frac{D}{E}\sin U\right)\sqrt{E}\,du$$

$$+\left(\cos V\frac{D''F-D'G}{GH}+\frac{D''}{G}\sin V\right)\sqrt{G}dv,$$

$$=(t\cos U+n\sin U)\sqrt{E}\,du+(t'\cos V+n'\sin V)\sqrt{G}\,dv.$$

Similarly, we find

$$\partial_2=(t\sin U-n\cos U)\sqrt{E}du+(t'\sin V-n'\cos V)\sqrt{G}\,dv,$$

$$\partial_3=-\frac{\partial U}{\partial u}du-\frac{\partial V}{\partial v}dv+\frac{\partial\omega}{dv}dv$$

$$+\frac{1}{H}\left\{\frac{\partial F}{\partial u}du-\tfrac{1}{2}\frac{\partial E}{\partial v}du+\tfrac{1}{2}\frac{\partial G}{\partial u}dv-\tfrac{1}{2}\frac{F}{E}\left(\frac{\partial E}{\partial u}du+\frac{\partial E}{\partial v}dv\right)\right\}$$

$$=\left(-\frac{1}{\sqrt{E}}\frac{\partial U}{\partial u}+g\right)\sqrt{E}\,du+\left(-\frac{1}{\sqrt{G}}\frac{\partial V}{\partial v}+g'\right)\sqrt{G}\,dv.$$

If we write now

$$p=\sqrt{E}(t\cos U+n\sin U),\quad q=\sqrt{E}(t\sin U-n\cos U),\quad r=-\frac{\partial U}{\partial u}+g\sqrt{E},$$

$$p'=\sqrt{G}\,(t'\cos V+n'\sin V),\, q'=\sqrt{G}(t'\sin V-n'\cos V),\, r'=-\frac{\partial V}{\partial v}+g'\sqrt{G},$$

$$\xi=\sqrt{E}\cos U,\quad \eta=\sqrt{E}\sin U,\quad \xi'=\sqrt{G}\cos V,\quad \eta'=\sqrt{G}\sin V,$$

we obtain, for the absolute displacements of a point in space,

$$\Delta x=dx+\xi du+\xi'dv-y(rdu+r'dv)+z(qdu+q'dv)$$

$$\Delta y=dy+\eta du+\eta'dv-z(pdu+p'dv)+x(rdu+r'dv)$$

$$\Delta z=dz-x(qdu+q'dv)+y(pdu+p'dv),$$

with the following relations

$$E=\xi^2+\eta^2,\quad F=\xi\xi'+\eta\eta',\quad G=\xi'^2+\eta'^2$$

$$D=p\eta-q\xi,\quad D'=p'\eta-q'\xi=p\eta'-q\xi',\quad D''=p'\eta'-q'\xi'.$$

This is the system of equations proved in Darboux, Vol. II, p. 348.

§ 55. Following Cesaro, or Eisenhart, p. 170, we may obtain the conditions satisfied by D, D', D″ due to Gauss and Codazzi.

If (x, y, z) is a point fixed in space, we have

$$\frac{\partial x}{\partial u}=-\xi+yr-zq,\quad \frac{\partial x}{\partial v}=-\xi'+yr'-zq',$$

$$\frac{\partial y}{\partial u}=-\eta+zp-xr,\quad \frac{\partial y}{\partial v}=-\eta'+zp'-xr',$$

$$\frac{\partial z}{\partial u}=xq-yp,\qquad \frac{\partial z}{\partial v}=xq'-yp'.$$

The above must be identically true for an infinite number of values of x, y, z as functions of u, v. Forming $\frac{\partial^2 x}{\partial u \partial v}$ we obtain

$$-\frac{\partial \xi}{\partial v}+y\frac{\partial r}{\partial v}-z\frac{\partial q}{\partial v}+r(-\eta'+zp'-xr')-q(q'x-p'y)$$

$$=-\frac{\partial \xi'}{\partial u}+y\frac{\partial r'}{\partial u}-z\frac{\partial q'}{\partial u}+r'(-\eta+pz-rx)-q'(qx-py)$$

This must be identically true and therefore

$$\frac{\partial \xi}{\partial v}-\frac{\partial \xi'}{\partial u}=\eta r'-\eta' r$$

$$\frac{\partial r}{\partial v}-\frac{\partial r'}{\partial u}=pq'-p'q$$

$$\frac{\partial q}{\partial v}-\frac{\partial q'}{\partial u}=rp'-r'p.$$

Similarly, from $\frac{\partial^2 y}{\partial u\,\partial v}$ and $\frac{\partial^2 z}{\partial u\,\partial v}$, we obtain

$$\frac{\partial \eta}{\partial v}-\frac{\partial \eta'}{\partial u}=\xi' r-\xi r',$$

$$\frac{\partial p}{\partial v}-\frac{\partial p'}{\partial u}=qr'-q'r,$$

$$\eta' p-\eta p'+\xi q'-\xi' q=0.$$

The first and fourth of these six equations may be verified without difficulty to be identically true, and the last equation is simply $D'-D'=0$. The other three equations will give the required relations.

Consider $\frac{\partial r}{\partial v}-\frac{\partial r'}{\partial u}=pq'-p'q$, substituting the values of p, p', q, q', r, r' we get

$$-\frac{\partial^2 U}{\partial u\,\partial v}+\frac{\partial^2 V}{\partial u\,\partial v}+\frac{\partial}{\partial v}\left\{\frac{1}{H}\left(\frac{\partial F}{\partial u}-\tfrac{1}{2}\frac{F}{E}\frac{\partial E}{\partial u}-\tfrac{1}{2}\frac{\partial E}{\partial v}\right\}\right.$$

$$\left.+\frac{\partial}{\partial u}\left\{\frac{1}{H}\left(\frac{\partial F}{\partial v}-\tfrac{1}{2}\frac{F}{G}\frac{\partial G}{\partial v}-\tfrac{1}{2}\frac{\partial G}{\partial u}\right)\right\}\right.$$

$$=\sqrt{EG}\ [tt'\sin(V-U)+nn'\sin(V-U)-(tn'-t'n)\cos(V-U)]$$

The right hand side is equal to

$$\frac{FD''-GD'}{HG}\cdot\frac{ED'-FD}{E}+\frac{DD''}{EG}-F\left\{\frac{ED'-FD}{HE}\frac{D''}{G}-\frac{FD''-GD'}{HG}\frac{D}{E}\right\}$$

$$=\frac{1}{H}(DD''-D'^2).$$

Also

$$\frac{\partial^2(V-U)}{\partial u\,\partial v}=\frac{\partial^2 \omega}{\partial u\,\partial v}=\frac{\partial}{\partial v}\left\{\frac{1}{H}\left(-\frac{\partial F}{\partial u}+\tfrac{1}{2}\frac{F}{E}\frac{\partial E}{\partial u}+\tfrac{1}{2}\frac{F}{G}\frac{\partial G}{\partial u}\right)\right\},$$

whence we have Gauss' equation :

$$\frac{DD''-D'^2}{H}=\frac{\partial}{\partial u}\left\{\frac{1}{H}\left(\frac{\partial F}{\partial v}-\frac{1}{2}\frac{F}{G}\frac{\partial G}{\partial v}-\frac{1}{2}\frac{\partial G}{\partial u}\right)\right\}$$
$$+\frac{\partial}{\partial v}\left\{\frac{1}{H}\left(\frac{1}{2}\frac{F}{G}\frac{\partial G}{\partial u}-\frac{1}{2}\frac{\partial E}{\partial v}\right)\right\}.$$

§ 56. *To obtain the equations of Codazzi from the relations*

$$\frac{\partial p}{\partial v}-\frac{\partial p'}{\partial u}=qr'-q'r,\ \frac{\partial q}{\partial v}-\frac{\partial q'}{\partial u}=rp'-r'p.$$

It will be found that, if we use the general equations, we have to multiply these equations by sin U, cos U and subtract. It is simpler to choose a special set of axes.

Take $U=\frac{\pi}{2}$, $V=\omega+\frac{\pi}{2}$ and substitute in $\frac{\partial p}{\partial v}-\frac{\partial p'}{\partial u}=qr'-q'r$. We find after considerable reduction,

$$\sqrt{E}\left\{\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{E}}\right)-\frac{\partial}{\partial u}\left(\frac{D'}{\sqrt{E}}\right)\right\}=\left\{\frac{(D'E-DF)}{H^2}\left(\frac{1}{2}\frac{\partial G}{\partial u}-\frac{1}{2}\frac{F}{E}\frac{\partial E}{\partial v}\right)+\right.$$
$$\left.\frac{(D'F-D''E)}{H^2}\left(\frac{\partial F}{\partial u}-\frac{1}{2}\frac{F}{E}\frac{\partial E}{\partial u}-\frac{1}{2}\frac{\partial E}{\partial v}\right)\right\};$$

or, finally

$$\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}=\frac{D}{H^2}\left(\frac{1}{2}G\frac{\partial E}{\partial v}-\frac{1}{2}F\frac{\partial G}{\partial u}\right)+\frac{D''}{H^2}\left(-E\frac{\partial F}{\partial u}+\frac{1}{2}F\frac{\partial E}{\partial u}+\frac{1}{2}E\frac{\partial E}{\partial v}\right)$$
$$+\frac{D'}{H^2}\left(\frac{1}{2}E\frac{\partial G}{\partial u}-\frac{1}{2}F\frac{\partial E}{\partial v}+F\frac{\partial F}{\partial u}-\frac{1}{2}F\frac{\partial E}{\partial v}-\frac{1}{2}G\frac{\partial E}{\partial u}\right).$$

Again, writing $U=\frac{\pi}{2}-\omega, V=\frac{\pi}{2}$, we obtain the second Codazzi equation which is derived from the former equation by interchanging everywhere u and v, E and G, D and D″.

We may notice the simplest forms of these equations when the parametric lines are the lines of curvature *viz.*, $F=0$, $D'=0$. They are:

$$DD''=-\tfrac{1}{2}\sqrt{EG}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}\right)\right],$$
$$EG\frac{\partial D}{\partial v}=\tfrac{1}{2}ED''\frac{\partial E}{\partial v}+\tfrac{1}{2}GD\frac{\partial E}{\partial u},$$
$$EG\frac{\partial D''}{\partial u}=\tfrac{1}{2}GD\frac{\partial G}{\partial u}+\tfrac{1}{2}ED''\frac{\partial G}{\partial u}.$$

Conversely, when D, D′, D″ satisfy these three equations there exists a surface, unique except for its position in space, having the six quantities E, F, G, D, D′, D″. (Compare: Eisenhart, § 65.)

§ 57. *The lines of curvature.*

The consecutive normals intersect in a centre of curvature,

$$\therefore \qquad \frac{x}{0}=\frac{y}{0}=\frac{z}{1}=\mathrm{R},$$

$$\frac{x-\xi}{\partial_2}=\frac{y-\eta}{-\partial_1}=\frac{z}{1}=\mathrm{R}\,;$$

whence $\xi\partial_1+\eta\partial_2=0$, which gives

$$(\mathrm{E}du+\mathrm{F}dv)(\mathrm{D}'du+\mathrm{D}''dv)-(\mathrm{F}du+\mathrm{G}dv)(\mathrm{D}du+\mathrm{D}'dv)=0,$$

determining the directions of the lines of curvature. This is, of course, simply $\mathrm{T}=0$, in § 51.

Again $\qquad \xi+\mathrm{R}\partial_2=0, \quad \eta-\mathrm{R}\partial_1=0.$

Multiply by cosV, sinV and add; and we find

$$\sqrt{\mathrm{E}}du\cos\omega+\sqrt{\mathrm{G}}dv-\frac{\mathrm{R}}{\sqrt{\mathrm{G}}}(\mathrm{D}'du+\mathrm{D}''dv)=0,$$

or

$$\mathrm{F}du+\mathrm{G}dv-\mathrm{R}(\mathrm{D}'du+\mathrm{D}''dv)=0.$$

Similarly, $\quad \mathrm{E}\,du+\mathrm{F}\,dv-\mathrm{R}\,(\mathrm{D}\,du+\mathrm{D}'\,dv)=0.$

Eliminate $du : dv$ and we have the equation for the radii of curvature,

$$\mathrm{R}^2\,(\mathrm{DD}''-\mathrm{D}'^2)-\mathrm{R}\,(\mathrm{ED}''+\mathrm{GD}-2\,\mathrm{FD}')+\mathrm{EG}-\mathrm{F}^2=0.$$

§ 58. By a suitable choice of co-ordinate parameters, we may have

(1) $\mathrm{F}=0$, $\mathrm{D}'=0$, the lines of curvature being parametric,

(2) $\mathrm{D}=0$, $\mathrm{D}''=0$, the asymptotic lines being parametric,

(3) $\mathrm{E}=0$, $\mathrm{G}=0$, the minimal lines being parametric.

This last system is of importance in discussing minimal surfaces.

We may notice that in the second case the parametric lines are not necessarily real and in the third case never so.

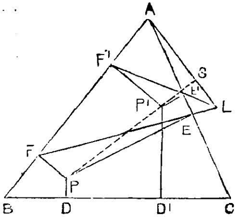
However when the parameters u, v are imaginary, the surface can still be real and all our results which are independent of any system of axes must hold good, since they could be derived by direct transformations.

Thus the various results for normal curvature, geodesic curvature and torsion hold good even when the parameters are imaginary.

SHORT NOTES.

The Pedal Circle.

Let DEF and D′E′F′ be the pedal triangles of isogonal conjugate points P and P′ with respect to a triangle ABC. EF, E′F′ intersect in L; FD, F′D′ in M, and DE, D′E′ in N. To prove by elementary geometry that AL, BM, CN are perpendicular to PP′.



Let S be the common point of intersection of the circumcircles of the triangles AFE, AF′E′, LEE′, LFF′. Since AE.AE′ = AF. AF′, A lies on SL, the radical axis of the circumcircles of LEE′, LFF′. Because the angles PEA and PFA are right angles, AP is a circum-diameter of AFE. Therefore PSA is a right angle, as also P′SA for a similar reason. Hence S lies an PP′ and AL is perpendicular to PP′. Similarly BM and CN are perpendicular to PP′.

If L′, M′, N′, be the points of intersection of EF′, E′F; FD′, F′D; D′E, D″E; then L′, M′, N′ lie on PP′. For ALL′ is self conjugate with respect to the pedal circle, being the harmonic triangle of the quadrilateral F′FE′E. Since AL is perpendicular to PP′, its pole L′ lies on PP′. Similarly, M′, N′ lie on PP.′

13th April, 1912. M. BHIMASENA RAO.

The Face of the Sky for September and October.

(The time referred to is Standard time).

Sidereal time at 8 p.m.

					September.					*October.*	
			H.	M.	S.		H.	M.	S.
1	...	...	18	41	35	...	20	39	55
8	...	...	19	9	11	...	21	7	31
15	...	...	19	36	47	...	21	35	7
22	...	...	20	4	23	...	22	2	43
29	...	...	20	31	59	...	22	30	19

From this table the constellations visible during the evenings of September and October can be ascertained by a reference to their position as given in a star-chart.

Phases of the Moon.

	D.	H.	M.			D.	H.	M.	
Last Quarter ...	6	9	48	A.M.	...	4	6	53	P.M.
New Moon ...	13	1	28	„	...	11	9	19	A.M.
First Quarter ...	19	10	27	P.M.	...	18	1	25	P.M.
Full Moon ...	28	1	29	A.M.	...	26	5	4	„

The Sun

Enters Libra on September 23, at 3-39 p.m.

The Planets.

Mercury begins as a morning star, attains its greatest elongation (17°58′W) on September 8, and is in superior conjunction with the Sun on October 4, when it becomes an evening star. It is in conjunction with the Moon on September 10 and October 11, with α Leonis on September 9, and with Mars on October 14.

Venus continues to be an evening star in these months. It is in conjunction with the Moon on September 12 at 6-31 p.m., and on October 12 at 9-36 p.m., and with Mars on September 9.

Mars is in conjunction with the Moon on September 12 and on October 11.

Jupiter is in conjunction with the Moon on September 17 and on October 14.

Saturn is stationary on September 16 when it begins to retrograde. It is in conjunction with the Moon on September 4 and October 1 at 0-52 a.m. and October 28 at 4-39 a.m.

Uranus, which was in opposition to the Sun on July 25, becomes stationary on October 9 at 8-30 p.m, when it resumes direct motion. The position of the planet at 8 p.m. is on

		R. A.				Decl.		
		H.	M.	S.		°	′	″
September 15	...	20	8	14·4	...	20	47	53·96
October 15	...	20	7	19·07	...	20	49	59·5

It is in conjunction with the Moon on September 21 and October 18.

Neptune, which was in conjunction with the Sun on July 16, is stationary on October 29 at 7-30 p.m. It is conjunction with the Moon on September 7 at 6-52 p.m. and on October 5 at 2-55 a.m. The position of the planet at 8 p.m. is on

		R. A.				Decl.		
		H.	M.	S.		°	′	″
September 15	...	7	49	26·46	...	20	33	15·79
October 15	...	7	51	25·94	...	20	28	2·54

Eclipses.

A Lunar eclipse occurs on September 26 and a Solar eclipse on October 10, both invisible in these parts.

V. Ramesam.

REVIEWS:

Geometry for Schools Vols. I—III: W. G. Borchardt and A. D. Perrott (London, G. Bell and Sons, 1912).

This is a useful volume covering the ground of the first two books of Euclid. The strictly deductive proofs of the more fundamental propositions are given in 26 Theorems and 13 Constructions of stage III. But the substance of almost all the remaining propositions with their ordinary modern extensions is given in the supplementary Theorems A—X; and a few important Problems, including Euc. II, 11 and 14, are discussed as Addenda. A large number of carefully chosen, practical exercises and 55 well-framed Test Papers enhance the value of the book.

In accordance with the Board of Education Circular 711 of 1909, an introduction to the first principles of the science by means of experimental work and deduction of facts therefrom is attempted in Stages I and II. The attempt is successful, the ideas of *surface, length and dimension* being clearly and concisely brought out, and the deduction of *facts* satisfactorily led up to.

The practical treatment of Areas should be expanded and placed, along with the deduction of a few *facts*, in the preliminary stages. The treatment of Loci is rather meagre; it might be expanded advantageously, and illustrated with a few more figures. The short treatment of Heights and Distances is good. We notice that two of the three diagrams in this section are *not* drawn to the scales mentioned in the problems solved.

In the definitions prefixed to stage III., the phrase "le theoreme contraire" is employed: we should prefer the English word *contrary*, used substantively like *converse*; it is really the *contrapositive* of the converse. The authors make no mention of the contrapositive of the original theorem, which is, of course, true or false along with the theorem.

Experimental and Theoretical Course of Geometry: A. T. Warren, (Oxford, Clarendon Press, 1911).

This course traverses the usual field of elementary geometry. The experimental part, about three-sevenths of the work, deals concisely but thoroughly with almost all parts of the subject. The treatment of Measurement, Folding of Paper and Symmetry is excellent. The author begins by defining *point* and *line* in Euclidean fashion; he gives no clear idea of *surface*, and does not refer to the forms of any *solid*

till towards the end of this part. This, we are afraid, is contrary to sound educational principles, at least in so far as the work claims to be elementary.

The theoretical part, comprising the remaining four-sevenths of the work, gives all the most important elementary properties and constructions of plane figures. The matter is skilfully arranged, and the proofs are distinctly above the average for brevity and lucidity. The author devotes a section to *Reductio ad Absurdum*, where some converse properties of circles and advanced ones of the triangle are proved. Otherwise this method of proof has not been employed.

Numerous practical exercises are given in the first part: in the second, there are about 400 exercises, all rather easy and a considerable number of them numerical.

Specimen Examination Papers occupying more than 30 pages are given at the end; they are mostly taken from the London Matriculation, Oxford and Cambridge School Examination, and Naval Cadets examination. For all these examinations, as also for the Cambridge Previous examination and Local Juniors' examination the book is, evidently, very suitable.

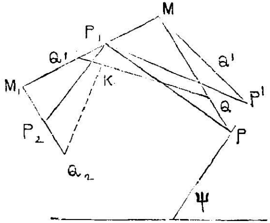
1st May, 1912. K. J. S.

SOLUTIONS.

Question 203.

(M. T. Naraniengar, M.A.) :—Through P a variable point on a given curve a straight line PQ is drawn cutting the curve at a constant angle. PQ = a definite function of the radius of curvature at P. Give a geometrical construction for the centre and radius of curvature at Q, as P moves on the curve.

Solution by the Proposer.



Let the constant angle at which PQ cuts the curve be α, and let $PQ = f(\rho) = \sigma$, say. Take P′Q′ a consecutive position of PQ, cutting PQ in M. Then, if P_1 be the centre of curvature at P, PP_1MP' is cyclic; and M is, in the limit, the foot of the perpendicular from P_1 on PQ.

(1) Now, produce MP_1 to Q_1 such that $P_1Q_1 = d\sigma/d\psi \equiv \sigma_1$, say. We shall prove that QQ_1 is *normal* to the locus of Q.

For, by projection, we easily get (since $QMQ' = \delta\psi$)

$$\tan Q'QM = \frac{PP'\sin\alpha - P'Q'\sin\delta\psi}{PP'\cos\alpha + P'Q'\cos\delta\psi - PQ} = \frac{\delta s\sin\alpha - (\sigma+\delta\sigma)\sin\delta\psi}{\delta s\cos\alpha + (\sigma+\delta\sigma)\cos\delta\psi - \sigma}$$

$$= \frac{\rho\sin\alpha - \sigma}{\rho\cos\alpha + \sigma_1} = \frac{PM - PQ}{MP_1 + P_1Q_1} = \frac{QM}{MQ_1},$$

so that $Q'QM = QQ_1M$, and therefore $Q'QQ_1$ is a right angle.

(2) Again, let P_2 be the centre of curvature of the evolute at P_1 and Q_2 a point on the perpendicular P_2M_1, such that $P_2Q_2 = d\sigma_1/d\psi = \sigma_2$. Then, denoting QQ_1M by λ, and the inclination of QQ′ to the fixed axis by ϕ, we have

$$\sin\lambda\,(\rho\cos\alpha + \sigma_1) = \cos\lambda\,(\rho\sin\alpha - \sigma) \text{ by (1).}$$

Differentiating with respect to ψ

$$\sin\lambda(\rho_1\cos\alpha + \sigma_2) + \cos\lambda(\rho\cos\alpha + \sigma_1)\frac{d\lambda}{d\psi} = \cos\lambda(\rho_1\sin\alpha - \sigma_1)$$

$$-\sin\lambda(\rho\sin\alpha - \sigma)\frac{d\lambda}{d\psi}.$$

$$i.e., \frac{d\lambda}{d\psi}\{\rho\cos(\alpha-\lambda)+\sigma_1\cos\lambda-\sigma_2\sin\lambda\}+$$

$$\{\sigma_1\cos\lambda+\sigma_2\sin\lambda-\rho_1\sin(\alpha-\lambda)\}=0$$

$$i.e., \frac{d\lambda}{d\psi}\{\text{sum of projections of } QP,\ PP_1,\ P_1Q_1 \text{ on } QQ_1\}$$

$$+\{\text{sum of projections of } Q_1P_1,\ P_1P_2,\ P_2Q_2\}=0$$

$$i.e., \frac{d\lambda}{d\psi}\cdot QQ_1+Q_1K=0.$$

Now, $\lambda+\phi=\alpha+\psi$

$\therefore$ $d\lambda+d\phi=d\psi$

$\therefore$ $\dfrac{d\lambda}{d\psi}+\dfrac{d\phi}{d\psi}=1$

$\therefore$ $\dfrac{d\phi}{d\psi}=1-\dfrac{d\lambda}{d\psi}=1+\dfrac{Q_1K}{Q_1}=m$, say.

Hence, the radius of curvature of the locus of Q is R such that

$$R=Lt\frac{QQ'}{\delta\phi}=Lt\frac{QQ'}{\delta\psi}\cdot\frac{\delta\psi}{\delta\phi}=Lt\frac{QQ'}{Q\hat{M}Q}\cdot Lt\frac{\delta\psi}{\delta\phi}=QQ_1\cdot\frac{d\psi}{d\phi}=QQ_1/m,$$

since QQ_1MQ' is cyclic in the limit.

Thus a geometrical construction for the centre of curvature is readily obtained.

Question 222.

(M. BHIMASENA RAO) :—$A_1\ B_1\ C_1$ is the pedal triangle of P with respect to a triangle ABC ; $A_2\ B_2\ C_2$ is the pedal triangle of P with respect to $A_1\ B_1\ C_1$, and so on. Show that (1) if $A_nB_nC_n$ is in perspective with $A_{n+1}\ B_{n+1}\ C_{n+1}$, it is also in perspective with $A_{n+2}\ B_{n+2}\ C_{n+2}$, and will not, generally, be in perspective with $A_{n+3}\ B_{n+3}\ C_{n+3}$; (2) if $A_n\ B_n\ C_n$ is in perspective with $A_{n+3}\ B_{n+3}\ C_{n+3}$, the pedal circle of P with respect to any one of the series of triangles touches the nine-point circle of that triangle.

Solution by S. Purushottam and the Proposer.

Let the angles PAB, PBC, PCA, PAC, PBA, PCB be denoted by $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$.

Then in the 1st pedal $\triangle$ angles PA_1B_1, etc., are.....$\gamma, \alpha, \beta, \beta', \gamma', \alpha'$;

" in the 2nd pedal..$\beta, \gamma, \alpha, \gamma', \alpha', \beta'$;

" in the 3rd pedal..$\alpha, \beta, \gamma', \alpha', \beta', \gamma'$;

Thus the 3rd pedal is similar to the original triangle, and the angle of similitude is found to be $90°-(\alpha+\beta+\gamma)$.

For the inverse triangle of ABC with respect to P, these angles are β', γ', α', γ, α, β; so that, the inverse triangle is similar to the 1st pedal triangle; the angle of similitude being $90^\circ-(\alpha+\beta+\gamma)$.

For the polar triangle of ABC with respect to P, the angles are γ', α', β', β, γ, α, since the polar triangle is the inverse of the 1st pedal triangle.

The above shows that when the geometrical relation between a point and a triangle can be expressed in the form $\phi(f, f')=0$, where ϕ is a symmetrical function of f, f' and f, f' are symmetrical functions of (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively, the point bears the same geometrical relation to the series of pedal, inverse and polar triangles of the original triangle.

(1) If ABC, and $A_1B_1C_1$ are in perspective,

$$\Pi\left(\frac{BA_1}{A_1C}\right)=1\,;\;\therefore\;\Pi\,(\cot\beta)=\Pi(\cot\gamma').$$

Also $\sin\alpha.\sin\beta.\sin\gamma=\sin\alpha'.\sin\beta'.\sin\gamma'$; therefore we have

$$\cos\alpha.\cos\beta.\cos\gamma=\cos\alpha'.\cos\beta'.\cos\gamma',$$

a relation of the form stated above.

Hence $A_1B_1C_1$ is in perspective with $A_2B_2C_2$; $A_2B_2C_2$ with $A_3B_3C_3$ and so on.

Now if DEF be any triangle inscribed in ABC and LMN be another triangle inscribed in DEF, and if P, Q, R be the points of intersection of AL, BM, CN with the sides of ABC, we have

$$\Pi\left(\frac{BD}{DC}\right).\Pi\left(\frac{BP}{PC}\right).\Pi\left(\frac{EL}{LF}\right)=1.$$

Thus if any one of the triangles ABC, DEF, LMN is in perspective with each of the remaining two, the latter two triangles are also in perspective.

Hence, if any one of the series of the pedal triangles is in perspective with its 1st pedal triangle, not only will every two consecutive triangles of the series be in perspective but each triangle will also be in perspective with its 2nd pedal triangle.

(2) A triangle and its 3rd pedal are similar; so that, if they are in perspective, they are homothetic. Therefore the angle of similitude $90^\circ-(\alpha+\beta+\gamma)$ vanishes:

$$\therefore\quad \alpha+\beta+\gamma=90^\circ.$$

If the ABC—normal c's of P be x, y, z, we have

$$\tan\alpha=\frac{z\sin A}{y+z\cos A},\quad \tan\beta=\frac{x\sin B}{z+x\cos B},\quad \tan\gamma=\frac{y\sin C}{x+y\cos C}.$$

But $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$.

$$\therefore\quad \Sigma\,\{zx\sin A\sin B\,(x+y\cos C)\}=\Pi\,(y+z\cos A)$$

$\therefore$ $\Sigma\, x \cos A\, (y^2 - z^2) = 0$, on simplification.

This represents the locus of a point whose pedal circle touches the nine point circle (see page 31 of Vol. II of the Journal). Hence, in this case, the pedal circle of P with respect to any one of the series of pedal, inverse and polar triangles touches the nine-point circle of that triangle.

Question 257.

(UNITY) :—If two conics have double contact with one another, then every confocal of one has double contact with some one confocal of the other, and the four common tangents to any confocal with the first and any confocal with the second all envelope a circle.

Solution by M. Bhimasena Rao.

Let B and C be the points of contact of the two conics and A be the point of intersection of the tangents to the conics at B and C. Taking ABC as the triangle of reference, let the tangential equations of the conic be

$$\beta\gamma - \kappa\alpha^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and

$$\beta\gamma - \kappa'\alpha^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Any confocal with (1) is $\beta\gamma - k\alpha^2 + \lambda\, (\Sigma\alpha^2 - 2\, \Sigma\beta\gamma \cos A) = 0 \ldots (3)$

Any confocal with (2) is $\beta\gamma - k'\alpha^2 + \lambda'\, (\Sigma\alpha^2 - 2\, \Sigma\beta\gamma \cos A) = 0 \ldots (4)$

If $\lambda = \lambda'$ (3) and (4) have double contact with one another, the pole of the chord of contact being $\alpha = 0$, *i.e.*, the point A.

Subtracting (4) from (3), we have

$$\alpha^2(\kappa - \kappa') + (\lambda - \lambda')(\Sigma\alpha^2 - 2\Sigma\beta\gamma\cos A) = 0,$$

which is a circle touching the common tangents of (3) and (4) and having its centre at $\alpha = 0$, *i.e.*, at A.

Additional Solution by S. Purushottam.

Question 294.

(S. RAMANUJAM) :—Shew that

$$\frac{e^x}{2} = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \ \ldots\ \frac{x^x}{\underline{|x}}\theta,$$

where θ lies between $\frac{1}{2}$ and $\frac{1}{3}$.

Solution by the Proposer.

$$\int_0^\infty e^{-z}\left(1 + \frac{z}{x}\right)^x dz = \int_0^\infty e^{-z} \left\{ \left(\frac{z}{x}\right)^x + \frac{x}{\underline{|1}}\cdot\left(\frac{z}{x}\right)^{x-1} + \frac{x(x-1)}{\underline{|2}}\left(\frac{z}{x}\right)^{x-2} + \&c., \right\} dz$$

$$=\frac{\underline{|x}}{x^x}\left(1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ \ldots\ldots +\frac{x^x}{\underline{|x}}\right).$$

Now, if we suppose that y is a function of x such that

$$\frac{e^x}{2}=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ \ldots\ldots\ldots +\frac{x^{x-1}}{\underline{|x-1}}+\frac{x^x}{\underline{|x}}y,$$

we see that

$$y=\frac{e^x\underline{|x}}{2x^x}-\frac{\underline{|x}}{x^x}\left(1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}\ \ \ldots\ldots\ldots +\frac{x^{x-1}}{x-1}\right)$$

$$=\frac{e^x\underline{|x}}{2x^x}+1-\int_0^\infty e^{-z}\left(1+\frac{z}{x}\right)^x dz,$$

$$=\frac{e^x\underline{|x}}{2x^x}+1-\int_0^\infty e^{-z}e^{z-\frac{z^2}{2x}+\frac{z^3}{3x^2}-\&c.}\ dz,$$

$$=\frac{e^x\underline{|x}}{2x^x}+1-\int_0^\infty e^{-\frac{z^2}{2x}+\frac{z^3}{3x^2}-\&c.,}\ dz,$$

$$=\frac{e^x\underline{|x}}{2x^x}+1-\sqrt{2x}\int_0^\infty e^{-z^2+\frac{2\sqrt{2}}{3\sqrt{x}}z^3-\&c.,}\ dz,$$

$$=\frac{z^x\underline{|x}}{2x^x}+1-\int_0^\infty e^{-z^2(\sqrt{2x}+\frac{4}{3}z^3-\&c.)}dz.$$

Therefore, when $x=\infty$,

$$y=\underset{x=\infty}{\mathrm{L}_t}\left\{\frac{e^x\underline{|x}}{2x^x}-\sqrt{\frac{\pi x}{2}}+1-\int_0^\infty e^{-z^2}\left(\tfrac{4}{3}z^3+\text{terms containing }\frac{1}{\sqrt{x}},\frac{1}{x}\&c.,\right)dz\right\}$$

$$=1-\frac{2}{3}=\frac{1}{3},$$

since by Stirling's theorem $\frac{e^x\underline{|x}}{2x^x}-\sqrt{\frac{\pi x}{2}}$, vanishes when x is infinite. Also $y=\frac{1}{2}$, when $x=0$.

Though the limits of y are known to be $\frac{1}{2}$ and $\frac{1}{3}$ when x is 0 and ∞ respectively, yet it is difficult to prove that y lies between $\frac{1}{2}$ and $\frac{1}{3}$.

The expansion of y in ascending powers of $\frac{1}{x}$ (which can be found from the above integral) is

$$\frac{1}{3}+\frac{4}{135x}-\frac{8}{2835x^2}-\frac{16}{8505x^3}+\&c.\qquad \ldots\qquad \ldots$$

This result is important as it helps to express e^x between rational limits.

Question 345.

(V. V. SATYANARAYAN) :—P and Q are the isotomic conjugates of the ortho-centre and the symmedian point of a triangle ABC. If AQ, BQ CQ meet the opposite sides in D,E,F respectively, shew that PD, PE, PF are parallel to the symmedians of the triangle.

Solution (1) *by M. Bhimasena Rao,* (2) *J. C. Swaminarayan, M.A., and N. Sankara Rao.*

(1) Let A′B′C′ be the medial triangle of ABC, and A″B″C″ be the triangle whose medial triangle is ABC. It is known that P is the symmedian point of A″B″C″. The triangles ABC, A′B′C′, A″B″C″ are homothetic and any point of A′B′C′ is collinear with and midway between its correspondents of ABC and A″B″C″. If K and K′ be the symmedian points of ABC and A′B′C′, AK, A′K′ and A″ P are parallel and KK′ = K′P. Therefore AK and A″P meet BC in points equidistant from A′ ; that is, A″ P meets BC in the point D given in the question. Similarly, B″ P, C″ P which are parallel to BK and CK meet CA and AB in E and F respectively. Hence the result.

(2) The areal co-ordinates of the ortho-centre and the symmedian point are : $(a \sec A, b \sec B, c \sec C)$; (a^2, b^2, c^2).

Therefore P and Q are

$$(\cos A/a, \cos B/b, \cos C/c) \text{ and } (1/a^2, 1/b^2, 1/c^2).$$

The equation to AQ is $yb^2 = zc^2$.

$\therefore$ D is (o, c^2, b^2).

Equation to P D is, therefore,

$$\begin{vmatrix} x & y & z \\ \cot A & \cot B & \cot C \\ o & c^2 & b^2 \end{vmatrix} = 0.$$

i.e., $a\,x\,(b \cos B \cos C) — b^2\, y \cos A + c^2\, z \cos A = 0.$

The equation of the symmedian A K is $y\, c^2 = z\, b^2$.

These two straight lines are parallel if

$$\begin{vmatrix} a\,(b \cos B - c \cos C) & -b^2 \cos A & c^2 \cos A \\ o & -c^2 & b^2 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$$i.e., \quad \begin{vmatrix} -(b^2-c^2)\cos A & -b^2\cos A & c^2\cos A \\ 0 & -c^2 & b^2 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

which is easily seen to be true.

Additional solution by N. Sanjiva Rao.

Question 348.

(J. C. SWAMINARAYAN, M.A.) :—Two circles are described through a given point so as to touch a given circle of radius r externally and internally with radii α and β respectively. If the given point is at a distance δ from the centre of the given circle, shew that

$$\frac{1}{\alpha}-\frac{1}{\beta}=\frac{4r}{\delta^2-r^2}.$$

Solution (1) by N. Sanjiva Rao and (2) by the Proposer.

(1) The two circles must be so described as to touch each other at the given point.

Let D be the given point and O the centre of the given circle.

Let A and B be the centres of the two circles that are described.

$AD=\alpha$, $BD=\beta$, and $KO=r$.

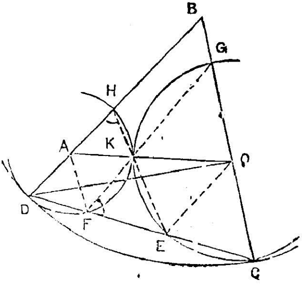
Join DC, cutting the circles at F and E.

The following are easily proved :

(i) FKG, HKE are collinear.

(ii) AF and EO are parallel repectively to BC and BD.

(iii) $\triangle^s$DHE and CFG are similar.



From (iii)

$$\frac{DE}{DH}=\frac{GC}{CF}$$

$$\therefore \quad CF=\frac{GC.DH}{DE}$$

$$\therefore \quad \frac{CF}{DF}=\frac{GC\cdot DH}{DF\cdot DE}=\frac{2GC}{DE}\cdot\frac{AD}{DF}$$

Because AF is parallel to BC,

$$\frac{AD}{DF}=\frac{BD}{DC}, \text{ and } \frac{CF}{DF}=\frac{BA}{AD}.$$

Hence $$\frac{BA}{AD}=\frac{2GC\cdot BD}{DE\cdot DC}$$

$$\therefore \quad \frac{BA}{AD.DB}=\frac{2\,GC}{DE\cdot DC}=\frac{2\,GC}{DO^2-OE^2}.$$

$$\therefore \quad \frac{\beta-\alpha}{\alpha\beta}=\frac{4r}{\delta^2-\gamma^2}\ldots$$

(2) Denote ODA by θ. Then

$$(\alpha+r)^2=\alpha^2+\delta^2-2\alpha\delta\cos\theta,$$

and $$(\beta-r)^2=\beta^2+\delta^2+2\beta\delta\cos\theta$$

$$\therefore \quad (\alpha+r)^2\beta-(\beta-r)^2\alpha=\alpha^2\beta-\beta^2\alpha+\delta^2(\alpha-\beta)$$

$$\therefore \quad 4\alpha\beta r=(\delta^2-r^2)/(\beta-\alpha)$$

$$\therefore \quad \frac{4r}{\delta^2-r^2}=\frac{\beta-\alpha}{\alpha\beta}$$

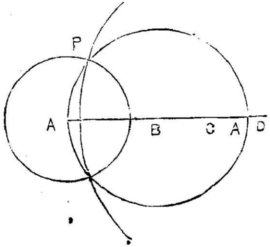
$$\therefore \quad \frac{1}{\alpha}-\frac{1}{\beta}=\frac{4r}{\delta^2-r^2}$$

Question 349.

(ZERO) :—A, B are two circles, such that the latter passes through the centre of the former. If C is any coaxal circle and D its inverse with respect to A, shew that the centres of C and D are inverse points with respect to B.

I. Solution by M. Bhimasena Rao, J. C. Swaminarayan, M.A. aud N. Sanjiva Rao.

Since C and D are inverses with respect to A, A, C, D, are co-axal and the centre of A is a centre of similitude of C and D. B belongs to the same co-axal system and passes through the centre of A; therefore it cuts the line of centres in the other centre of similitude of C and D and is their circle of similitude. Since the centres of similitude and the centres of C and D form a harmonic range, the two latter are inverse points with respect to the circle of similitude.



II. Analytical Solution by M. Satyanarayana, M. A. and J. C. Swaminarayan, M. A.

Let the centre of ⊙ A be taken as origin.

Then $x^2+y^2=a^2$, and $x^2+y^2=2\,bx$ will be the equations of ⊙s A and B respectively.

$\therefore \quad 2bx - a^2 = 0$, is the equation of the line of intersection.

The equation of C can be written in the form

$$x^2+y^2-a^2+\lambda\,(2bx-a^2)=0,$$

i.e.,
$$x^2+y^2+2\lambda bx-a^2\,(1+\lambda)=0.$$

The equation of ⊙ D will therefore be

$$\frac{a^4}{x^2+y^2}+\frac{2\lambda a^2bx}{x^2+y^2}-a^2\,(1+\lambda)=0.$$

i.e.,
$$a^2\,(1+\lambda)\,(x^2+y^2)-2\lambda a^2bx-a^4=0.$$

Thus $\quad AC = -\lambda b \quad AD = \dfrac{\lambda b}{1+\lambda}$

$\therefore \quad BC = b+\lambda b = b\,(1+\lambda)$

$$BD = b - \frac{\lambda b}{1+\lambda} = \frac{b}{1+\lambda}$$

$\therefore \quad BC \cdot BD = b^2.$

Hence, C and D are inverse points with respect to B.

Additional Solution by N. Sankara Aiyar.

Question 350.

(M. T. Naraniengar M.A.) :—G is the centroid of a triangle ABC. AG, BG, CG are produced to meet the circum-circle in D, E, F; shew that G is a focus of the maximum inscribed ellipse of DEF.

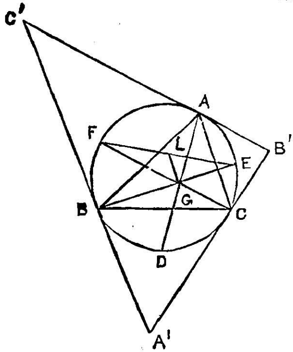
Solution (1) by M. Bhimasena Rao and (2) by 'zero.'

(1) Let A'B'C' be the triangle formed by lines through A, B, C at right angles to GA, GB, GC. G is the symmedian point of A'B'C'.

Reciprocate the circumcircle of A'B'C' with respect to the circle, centre G, and radius $=\sqrt{GA\cdot GD}=\sqrt{GB\cdot GE}=\sqrt{GC\cdot GF}$. The polar of D is B'C'.

We see therefore that the polars of A', B', C' are the sides of DEF and that the polar reciprocal is inscribed in DEF, having its focus at G. Since the tangent at A' to the circumcircle of A'B'C', and GA' are harmonic conjugates with respect to A'B' and A'C', the polar reciprocal touches EF at its middle point. Similarly the polar reciprocal touches FD, DE at their middle points, and is therefore the maximum inscribed ellipse of DEF.

(2) Suppose G is a focus of the maximum inscribed ellipse of DEF touching EF at its middle point L. Then, from the properties of the tangent and the focus, we have



$E\hat{G}D + L\hat{G}F = 2$ right angles.

$\therefore$ $B\hat{G}D = F\hat{G}L$ ... ... (1)

Now Δ^s BGC, FGE are similar, and GL bisects FE.

$\therefore$ GD bisects BC, by (1).

In other words, AG is a median of the Δ ABC. Similarly, for BG and CG.

Hence G is the centroid of ABC.

Analytical solution by J. C. Swaminarayan, M.A.

Question 355.

(N. SANKARA AIYAR) :—Solve completely

$$x(1+x)^2 \frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} + y = x.$$

Solution by S. Narayanan, B.A., L.T.

The equation $x(1+x)^2 \frac{d^2y}{dx^2} + x(1+x)\frac{dy}{dx} + y = x$, may be written

$$\left(x.\overline{1+x}\frac{d}{dx} + 1\right)\left(\overline{1+x}\frac{dy}{dx} - \frac{y}{x}\right) = x.$$

Let $\overline{1+x}\frac{dy}{dx} - \frac{y}{x} = v$, then

$$x.\overline{1+x}\frac{dv}{dx} + v = x.$$

Dividing by $(1+x)^2$ and integrating

$$v = \text{A}\ \frac{1+x}{x} + \frac{1}{x} + \frac{x+1}{x}\log(1+x).$$

$$\therefore \qquad \overline{1+x}\frac{dy}{dx} - \frac{y}{x} = \text{A}\ \frac{1+x}{x} + \frac{1}{x} + \frac{x+1}{x}\log(1+x).$$

Divide, again, by x and integrate :

$$y.\frac{x+1}{x} = \text{B} + \text{A}\left(\log x - \frac{1}{x}\right) - \frac{1}{x} + \int \frac{x+1\ \log(x+1)}{x^2}dx.$$

Question 360.

(J. C. SWAMINARAYAN, M.A.) :—G is the centroid of a triangle ABC and G′ that of the triangle formed by the images of any point P on the circumcircle in the sides of ABC. Shew that GG′ is perpendicular to the Simson's line of P.

Solution (1) by M. Bhimasena Rao and (2) by the Proposer.

(1) *First Proof*: Let L,M,N be the images of P. L,M,N lie on the reflections of the circumcircle of ABC in the sides. These reflections are the circles BHC, CHA, AHB, where H is the ortho-centric of ABC; their inverses with respect to the polar circle of ABC are the sides EF, FD, DE of the ortho-centric triangle of ABC. Also L,M,N,H are collinear; if the line of collinearity meets the sides of DEF in L′, M′, N′ and the axis of perspective of ABC and DEF in G″, we see that L′, M′, N′ are inverses of L, M, N with respect to the polar circle.

By the question HL+HM+HN=3 HG′.

From the property of the polar line,

$$\frac{1}{HL'}+\frac{1}{HM'}+\frac{1}{HN'}=\frac{3}{HG''}$$

Thus G″ is the inverse of G′, and hence G′ lies on the ortho-centroidal circle of ABC. For, it is the inverse of the axis of perspective of ABC and DEF with respect to the polar circle. Hence G G′ is perpendicular to H G′, etc.

Second Proof: Let A′B′C′ be the triangle formed by lines through ABC parallel to BC, AC, AB, and P the focus of the parabola described to ABC, having HLMN for its directrix. Therefore P is the orthopole of LMN with respect to A′B′C′. PL,PM,PN are perp. to the sides of A′B′C′. Therefore A′L, B′M, C′N are perp. to LMN. Hence the foot of the perp. from G on LMN is the mean-centre of L,M,N, etc.

(2) With P as focus describe a parabola touching BC, CA, AB at X,Y,Z, respectively.

The three images, then, lie on the directrix of the parabola and the Simson's line of P is the tangent at the vertex. Hence if ϕ_1, ϕ_2, ϕ_3 are the intrinsic angles of X,Y,Z, respectively, the co-ordinates of A,B,C are

$$(a\tan\phi_2\tan\phi_3),\ a(\tan\phi_2+\tan\phi_3)$$
$$(a\tan\phi_3\tan\phi_1),\ a(\tan\phi_3+\tan\phi_1)$$
$$(a\tan\phi_1\tan\phi_2),\ a(\tan\phi_1+\tan\phi_2)$$

The $\perp^r$ from P on BC, the tangent at P, is

$$(x-a)\tan\phi+y=0$$

and cuts directrix at a point whose ordinate is $2a\tan\phi_1$

The co-ordinates of the three images are

$$-a,\ 2a\tan\phi_1;\ -a,\ 2a\tan\phi_2;\ -a,\ 2a\tan\phi_3.$$

Thus the ordinate of G = the ordinate of G′,

$$=\tfrac{2}{3}a(\tan\phi_1+\tan\phi_2+\tan\phi_3).$$

Hence GG′ is $\perp$ to the axis of y that is $\perp^r$ to the Simson's line of P.

QUESTIONS FOR SOLUTION.

388. (M. V. ARUNACHALA, SASTRY, M.A.,) :—If $al+bm+cn=(a+b+c)p$, prove that $al^2+bm^2+cn^2>(a+b+c)p^2$; a,b,c being positive quantities.

389. (D. D. KAPADIA, M.A., B. SC.) :—A square $A_1\ A_2\ A_3\ A_4$ and a regular pentagon $B_1\ B_2\ B_3\ B_4\ B_5$ are inscribed in a circle of radius r. Shew that the continued product of all the chords AB is numerically equal to $2r^{20}$ sin 20θ, where 2θ, is the angle subtended by any one of these chords at the centre of the circle.

390. (M. V. ARUNACHALA SASTRY, M.A.,) :—A variable circle C touches a given stright line at a given point and cuts a given circle in two points P, Q. Find the locus of the centres of circles cutting the circle C orthogonally at P, Q.

391. (D. D. KAPADIA, M.A., B. SC.,) :—Prove that the sum of the reciprocals of the co-efficients in the expansion of $(1-x)^{2n}$ is $2-1/(n+1)$ where n is a positive integer.

392. (A. C. L. WILKINSON) :—If in Question 313 the circle TQQ′ be replaced by any conic though TQQ′ touching the ellipse at P; the radii of curvature of the two conics at P are in the ratio 1 : 2

393. (T. RAJARAMA RAO) :—A is the vartex of a triangle on a given base BC, having a given vertical angle. Find the locus of the isogonal conjugates of the triangle ABC that are equidistant from A.

394. (T. P. TRIVEDI, M.A., LL. B.) In a spherical triangle, if

$$x=\cot\frac{b}{2}\cot\frac{c}{2},\ y=\cot\frac{c}{2}\cot\frac{a}{2},\ z=\cot\frac{a}{2}\cot\frac{q}{2},$$

and
$$X=\tan\frac{B}{2}\tan\frac{C}{2},\ Y=\tan\frac{C}{2}\tan\frac{A}{2},\ Z=\tan\frac{A}{2}\tan\frac{B}{2},$$

then
$$(1-x)(Y-Z)=(1-X)(y-z),$$

with two similar equalities.

395. (K. J. SANJANA, M.A.)—Prove the identity

$$\Sigma a(b-c)\,(b+c-a)\,(b^p+c^p-a^p)=-2\Sigma a^p bc\,(b-c)$$
$$=2abc(b-c)(c-a)(a-b)\ S_{p-3}$$

when S_{p-3} stands for the sum of the homogeneous products of a,b,c, of $(p-3)$ dimensions; and find when the left side identically vanishes (Suggested by Q. 351 of Prof. J. C. Swaminarayan.)

396. (K. J. SANJANA and T. P. TRIVEDI.)—Prove the following identities :—

(i) $(n+1)n\sin\theta+n(n-1)\sin2\theta+(n-1)(n-2)\sin3\theta\ldots.+2\cdot1\sin n\theta.$

$$=\frac{1}{2}(n+1)(n+2)\cot\frac{\theta}{2}-\frac{1}{2}\operatorname{cosec}^3\frac{\theta}{2}\sin\frac{n+1}{2}\theta\sin\frac{n+2}{2}\theta;$$

(ii) $(n+1)n\sin n\theta+n(n-1)\sin(n-1)\theta+\ldots 3\cdot 2\sin 2\theta+2\cdot 1\sin\theta$

$$=\frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}\left\{(n+1)(n+2)\sin n\theta-(n+1)n\sin(n+1)\theta\right\}$$

$$-\frac{1}{2}\operatorname{cosec}^3\frac{\theta}{2}\sin\frac{n}{2}\theta\sin\frac{n+1}{2}\theta;$$

(iii) $(n-1)^3\sin\theta+(n-2)^3\sin 2\theta+\ldots\ldots 1^3\sin(n-1)\theta$

$$=\frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}\left\{n^3\sin\theta-\sin n\theta\right\}$$

$$-\frac{3}{8}\operatorname{cosec}^4\frac{\theta}{2}\left\{n\sin\theta-\sin n\theta\right\};$$

(iv) $(n-1)^4\cos\theta+(n-2)^4\cos 2\theta+\ldots\ldots 1^4\cos(n-1)\theta$

$$=-n^4+n^3\operatorname{cosec}^2\frac{\theta}{2}-\tfrac{1}{2}n\operatorname{cosec}^2\frac{\theta}{2}\left(3\operatorname{cosec}^2\frac{\theta}{2}-1\right)$$

$$+\frac{1}{8}\sin n\theta\sin\theta\operatorname{cosec}^4\frac{\theta}{2}\left(3\operatorname{cosec}^2\frac{\theta}{2}-1\right).$$

[The last two identities are suggested by Prof. Kapadia's Q. 383.]

397. (J. C. SWAMINARAYAN, M.A.) :—A particle moves under an attractive force $\frac{\mu m r}{(r^2+k^2)^2}$, and is projected from an apse at a distance a with a velocity equal to that acquired by falling from infinity to the apse under the action of the same force. Shew that it will describe a circle of radius $\frac{1}{2}(a+k^2/a)$ with its centre at a distance $\frac{1}{2}(a-k^2/a)$ from the centre of force.

398. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—A hyperboloid is described having for generators the edges AB, BC, CD, DA of a tetrahedron ABCD and with its centre at the centroid of the tetrahedron, show that the algebraical sum of the squares of its axes is equal to

$$\tfrac{1}{4}(3\,AC^2+3\,BD^2-AB^2-BC^2-CD^2-DA^2)$$

and that the absolute value of the product of the axes is three times the volume of the tetrahedron ABCD.

399. (P. V. SESHU AIYAR, B.A., L.T.) :—Solve completely the differential equation :—

$$c\left(x+y\frac{dy}{dx}\right)=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$